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Boundary blow-up elliptic problems with nonlinear gradient terms ☆

Zhijun Zhang

Department of Mathematics and Informational Science, Yantai University, Yantai, Shandong, 264005, PR China

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Abstract

By Karamata regular variation theory and perturbation method, we show the exact asymptotical behaviour of solutions near the boundary to nonlinear elliptic problems $\Delta u \pm |\nabla u|^q = b(x)g(u)$, $u > 0$ in Ω, $u|_{\partial\Omega} = +\infty$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $q \geqslant 0$, $g \in C^1[0,\infty)$, $g(0) = 0$, g' is regularly varying at infinity with index ρ with $\rho > 0$ and b is nonnegative nontrivial in Ω, which may be vanishing on the boundary.

1. Introduction and the main results

The purpose of this paper is to investigate the exact asymptotic behaviour of the solutions near the boundary to the following model problems

$$\Delta u \pm |\nabla u|^q = b(x)g(u), \quad u > 0 \text{ in } \Omega,\ u|_{\partial\Omega} = +\infty, \tag{P$_\pm$}$$

where the last condition means that $u(x) \to +\infty$ as $d(x) = \operatorname{dist}(x, \partial\Omega) \to 0$, and the solution is called "large solution" or "explosive solution," Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geqslant 1)$, $q \geqslant 0$, g satisfies

(g_1) $g \in C^1[0,\infty)$, $g(0) = 0$, g is increasing on $[0,\infty)$;

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E-mail address: zhangzj@ytu.edu.cn.

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(g_2) Keller–Osserman condition:

$$\int_t^\infty \frac{ds}{\sqrt{2G(s)}} < \infty, \quad \forall t > 0,\ G(s) = \int_0^s g(z)\,dz;$$

and b satisfies

(b_1) $b \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$, is nonnegative in Ω and has the property: if $x_0 \in \Omega$ and $b(x_0) = 0$, then there exists a domain Ω_0 such that $x_0 \in \Omega_0 \subset \Omega$ and $b(x) > 0, \forall x \in \partial\Omega_0$.

The main feature of this paper is the presence of the three terms, the nonlinear term $g(u)$ which is regularly varying at infinity with index $1+\rho$ and $\rho > 0$ and includes a large class of functions, the nonlinear gradient term $\pm|\nabla u|^q$ and the weight $b(x)$ which may be vanishing not only on large parts of Ω but also on the boundary and also includes a large class of functions.

First, let us review the following model

$$\Delta u = b(x)g(u) \quad \text{in } \Omega,\ u|_{\partial\Omega} = +\infty. \tag{P$_0$}$$

Problem (P_0) arises from many branches of mathematics and applied mathematics and have been discussed by many authors and in many contexts; see, for instance, [3–16,20–22,24,26–32, 34,35,38–40,42,43].

When $b \equiv 1$ on Ω:

Keller and Osserman [22,34] first supplied a necessary and sufficient condition (g_2) for the existence of large solutions to problem (P_0).

When $g(u) = e^u$, problem (P_0) was studied much earlier. Bieberbach (see [26]) proved that when $N = 2$, there is one solution $u \in C^2(\Omega)$ satisfying $|u(x) - \ln(d(x))^{-2}|$ is bounded on Ω. The same results was proved by Rademacher (see [26]) for $N = 3$.

Later, Loewner and Nirenberg [28] showed that if $g(u) = u^{p_0}$ with $p_0 = (N+2)/(N-2)$, $N > 2$, then problem (P_0) has a unique positive solution u satisfying

$$\lim_{d(x)\to 0} u(x)\big(d(x)\big)^{(N-2)/2} = \big(N(N-2)/4\big)^{(N-2)/4}.$$

Then, by analyzing the corresponding ordinary differential equation, combining with the maximum principle, Bandle and Marcus [3] established the following results: if g satisfies (g_1) and

(g_3) there exist $\theta > 0$ and $s_0 \geqslant 1$ such that $g(\xi s) \leqslant \xi^{1+\theta} g(s)$ for all $\xi \in (0,1)$ and $s \geqslant s_0/\xi$;

then for any solution u of problem (P_0)

$$\frac{u(x)}{\psi_1(d(x))} \to 1 \quad \text{as } d(x) \to 0, \tag{1.1}$$

where ψ_1 satisfies

$$\int_{\psi_1(t)}^\infty \frac{ds}{\sqrt{2G(s)}} = t, \quad \forall t > 0. \tag{1.2}$$

And, in addition to the conditions given above, g satisfies

(g_4) $g(\xi s) \leqslant \xi g(s)$ for all $\xi \in (0,1)$ and all $s \geqslant 0$; this is equivalent to that $g(s)/s$ is increasing on $(0,\infty)$; and so is that $g(\xi s) \geqslant \xi g(s)$ for all $\xi \geqslant 1$ and all $s \geqslant 0$;

then problem (P_0) has a unique solution.

Moreover, Lazer and McKenna [27] showed that if g satisfies (g_1) and

(g_5) there exists $a_1 > 0$ such that $g'(s)$ is nondecreasing for $s \geqslant a_1$, and

$$\lim_{s\to\infty} \frac{g'(s)}{\sqrt{G(s)}} = \infty,$$

then for any solution u of problem (P_0)

$$u(x) - \psi_1\big(d(x)\big) \to 0 \quad \text{as } d(x) \to 0. \tag{1.3}$$

When b satisfies (b_1), g satisfies (g_1) and (g_2), Lair [24] showed existence of solutions to problem (P_0).

Most recently, applying the regular varying theory, which were first introduced and established by Karamata in 1930 and is a basic tool in stochastic process, and constructing comparison functions, Cîrstea and Rădulescu [7–10], Cîrstea and Du [6], the author [43] showed the uniqueness and exact asymptotical behaviour of solutions near the boundary to problem (P_0). A basic result is that if g satisfies (g_1), (g_4) and

(g_6) there exists $\rho > 0$ such that $\lim_{s\to\infty} \frac{g'(\xi s)}{g'(s)} = \xi^\rho$, $\forall \xi > 0$,

and $b \in C^\alpha(\overline{\Omega})$, $b \geqslant 0$ in Ω, and satisfies

(b_2) $\lim_{d(x)\to 0+} \frac{b(x)}{k^2(d(x))} = c_0 > 0$ foe some $k \in K$ with $l_1 > 0$;

then any solution u of problem (P_0) satisfies

$$\lim_{d(x)\to 0} \frac{u(x)}{\psi(d(x))} = \xi_0, \tag{1.4}$$

where $\xi_0 = (\frac{2+l_1\rho}{c_0(2+\rho)})^{1/\rho}$, and $\psi \in C^2(0,a)$ ($a \in (0,\nu)$) is defined by

$$\int_{\psi(t)}^{\infty} \frac{ds}{\sqrt{2G(s)}} = \int_0^t k(s)\,ds, \quad \forall t \in (0,a); \tag{1.5}$$

K denotes the set of all positive nondecreasing C^1-functions k defined on $(0,\nu)$, which satisfy

$$\lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{k(t)} = 0 \quad \text{and} \quad \lim_{t\to 0^+} \frac{d}{dt}\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right) = l_1.$$

We note that for each $k \in K$, $l_1 \in [0, 1]$ and $\psi(d(x)) = \psi_1(\int_0^{d(x)} k(s)\,ds)$.

Now let us return to problem ($P_\pm$).

When $b \equiv 1$ on Ω: for $g(u) = u^p$, $p > 0$, by the ordinary differential equation theory and the comparison principle, Bandle and Giarrusso [2] showed the following results:

(i) if $p > 1$ or $q > 1$, then problem (P_+) has one solution in $C^2(\Omega)$; and the same statement is true to problem (P_-) if $p > 1$ or $1 < q \leqslant 2$;

(ii) if $p > 1$ and $0 < q < 2p/(p+1)$, then every solution $u_\pm$ to problem ($P_\pm$) satisfies (1.1), where $\psi_1(t) = (\sqrt{2(p+1)}/(p-1))^{2/(p-1)} t^{-2/(p-1)}$;

(iii) if $\frac{2p}{p+1} < q < p$, then for any solution u_+ to problem (P_+)

$$\lim_{d(x)\to 0} u_+(x)\left(\frac{p-q}{q} d(x)\right)^{q/(p-q)} = 1; \tag{1.6}$$

(iv) if $\max\{2p/(p+1), 1\} < q < 2$, then every solution u_- to problem (P_-) satisfies

$$\lim_{d(x)\to 0} u_-(x)(2-q)\big((q-1)d(x)\big)^{(2-q)/(q-1)} = 1; \tag{1.7}$$

(v) if $q = 2$, then every solution u_- to problem (P_-) satisfies

$$\lim_{d(x)\to 0} u_-(x)/-\ln\big(d(x)\big) = 1. \tag{1.8}$$

Moreover, Bandle and Giarrusso [2] extended the above results for more general $g(u)$ satisfying

$$\lim_{u\to\infty} \frac{\sqrt{G(u)}}{(g(u))^{1/q}} = 0 \quad \text{or} \quad \lim_{u\to\infty} \frac{\sqrt{G(u)}}{(g(u))^{1/q}} = \infty.$$

Then Giarrusso [17,18] showed the following results:

(vi) Let g satisfies (g_1) and $1 < q < 2$. If g satisfies

(g_6) $\lim_{u\to\infty} \frac{g(u)}{u^{q/(2-q)}} = \gamma_0 \in (0, \infty)$ (this is equivalent to that $\frac{\sqrt{G(u)}}{(g(u))^{1/q}}$ converges to a positive number as $u \to \infty$);

then for any solution $u_\pm$ to problem ($P_\pm$)

$$\lim_{d(x)\to 0} u_\pm(x)\left(\frac{q-1}{2-q}\sqrt{\zeta_0^\pm}\, d(x)\right)^{(2-q)/(q-1)} = 1, \tag{1.9}$$

where $\zeta_0^\pm$ is the unique positive root of the equation

$$\frac{\zeta}{2-q} \pm \zeta^{q/2} = \gamma_0. \tag{1.10}$$

Then, the author [44] considered problem ($P_\pm$) under the condition (ii) for weight b which may be singular on the boundary. Lasry and Lions [25] established existence, uniqueness and exact asymptotical behaviour of solutions near the boundary to problem (P_-) for $g(u) = \lambda u$ with

$\lambda > 0$ and $q > 1$. For other existence results of large solutions to elliptic problems with nonlinear gradient terms, see [41,45].

In this paper, also applying Karamata regular variation theory, perturbed method and constructing comparison functions, we show asymptotic behaviour of solutions near the boundary to problem ($P_\pm$).

First let us recall some basic definitions and the properties to Karamata regular variation theory [33,36,37].

Definition 1.1. A positive measurable function f defined on $[a,\infty)$, for some $a>0$, is called regularly varying at infinity with index ρ, written $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{f(\xi t)}{f(t)} = \xi^\rho. \tag{1.11}$$

The real number ρ is called the index of regular variation.

When $\rho = 0$, we have:

Definition 1.2. A positive measurable function L defined on $[a,\infty)$, for some $a>0$, is called slowly varying at infinity, if for each $\xi > 0$

$$\lim_{t\to\infty} \frac{L(\xi t)}{L(t)} = 1. \tag{1.12}$$

It follows by Definitions 1.1 and 1.2 that if $f \in RV_\rho$ it can be represented in the form

$$f(t) = u^\rho L(t). \tag{1.13}$$

Lemma 1.1 *(Uniform convergence theorem). If $f \in RV_\rho$, then* (1.11) (*and so* (1.12)) *holds uniformly for $\xi \in [a,b]$ with $0 < a < b$.*

Lemma 1.2 *(Representation theorem). The function L is slowly varying at infinity if and only if it may be written in the form*

$$L(u) = c(u)\exp\left(\int_a^u \frac{y(s)}{s}\,ds\right), \quad u \geqslant a, \tag{1.14}$$

for some $a > 0$, where $c(u)$ and $y(u)$ are measurable and for $u \to \infty$, $y(u) \to 0$ and $c(u) \to c$, with $c > 0$.

Definition 1.3. A positive measure function h defined on some neighborhood $(0,a)$ for some $a>0$, is called regularly varying at zero with index σ, written $h \in RVZ_\sigma$, if for each $\xi > 0$ and some $\sigma \in \mathbb{R}$,

$$\lim_{t\to 0^+} \frac{h(\xi t)}{h(t)} = \xi^\sigma. \tag{1.15}$$

Our main results are as the following.

Theorem 1.1. *Let g satisfies* (g_1), $g'(t) = u^\rho L(t)$, $\rho > 0$, *L is slowly varying at infinity, and* $1 < q < \rho + 1$, *b satisfies* (b_1) *with $b = 0$ on $\partial\Omega$ and*

(b_3) $\lim_{d(x)\to 0+} \frac{b(x)}{k^q(d(x))} = c_q > 0$ *for some $k \in K$ with $l_1 > 0$; $\varphi \in C^2(0, a)$ be uniquely determined by*

$$\int_{\varphi(t)}^{\infty} \frac{ds}{(g(s))^{1/q}} = \int_0^t k(s)\,ds, \quad \forall t \in (0, a). \tag{1.16}$$

(I) *If $q = \frac{2(\rho+1)}{\rho+2}$ and $\lim_{s\to\infty} L(s) = (1+\rho)\gamma_0 \in (0, \infty)$, then every solution $u_+ \in C^2(\Omega)$ to problem* (P_+) *satisfies*

$$\lim_{d(x)\to 0} \frac{u_+(x)}{\varphi(d(x))} = c_q^{-1/\rho-q+1}, \tag{1.17}$$

where

$$\varphi(t) = \left(\frac{2-q}{\gamma_0^{1/q}(q-1)}\right)^{(2-q)/(q-1)} \left(\int_0^t k(s)\,ds\right)^{-(2-q)/(q-1)}, \quad t \in (0, a).$$

(II) *The same statement is true if $\frac{2(\rho+1)}{\rho+2} < q \leqslant 2$, where $\varphi \in RVZ_{-q/l_1(\rho+1-q)}$ and there exists $H \in RVZ_0$ such that*

$$\varphi(t) = H(t)t^{-q/l_1(\rho+1-q)}. \tag{1.18}$$

Corollary 1.1. *Let $g(u) = u^{\rho+1}$, $\rho > 0$, $1 < q < \rho + 1$ and $b(x) \cong c_q(d(x))^{\tau/q}$ with $\tau > 0$ near $\partial\Omega$. If $\frac{2p}{p+1} \leqslant q \leqslant 2$, then $\gamma_0 = 1$, $l_1 = \frac{q}{q+\tau}$,*

$$\varphi(t) = \left(\frac{q+\tau}{\rho+1-q}\right)^{q/(\rho+1-q)} t^{-(q+\tau)/(\rho+1-q)}, \quad t > 0,$$

i.e., any solution u_+ to problem (P_+) *satisfies*

$$\lim_{d(x)\to 0} u_+(x)\big(d(x)\big)^{(q+\tau)/(\rho+1-q)} = c_q^{-1/\rho-q+1}\left(\frac{q+\tau}{\rho+1-q}\right)^{q/(\rho+1-q)}.$$

Theorem 1.2. *Let $b \equiv 1$ on Ω, g satisfies* (g_1), $g'(t) = u^\rho L(t)$, $\rho > 0$, *L is slowly varying at infinity, and $1 < q < \rho + 1$, $\varphi_1(t)$ be uniquely determined by*

$$\int_{\varphi_1(t)}^{\infty} \frac{ds}{(g(s))^{1/q}} = t, \quad \forall t > 0. \tag{1.19}$$

(I) *If* $q = \frac{2(\rho+1)}{\rho+2}$ *and* $\lim_{s\to\infty} L(s) = (1+\rho)\gamma_0 \in (0,\infty)$, *then every solution* $u_\pm \in C^2(\Omega)$ *to problem* $(\mathrm{P}_\pm)$ *satisfies*

$$\lim_{d(x)\to 0} \frac{u_\pm(x)}{\varphi_1(d(x))} = \xi_0^\pm, \tag{1.20}$$

where $\xi_0^\pm$ *is the unique positive solution to the equation*

$$\frac{\rho+1}{q}\gamma_0^{(2-q)/q} - \xi^\rho \pm \xi^{q-1} = 0 \tag{$q_\pm$}$$

and

$$\varphi_1(t) = \left(\frac{2-q}{\gamma_0^{1/q}(q-1)}\right)^{(2-q)/(q-1)} t^{-(2-q)/(q-1)}, \quad t > 0.$$

(II) *If* $q > \frac{2(\rho+1)}{\rho+2}$, *then every solution* $u_+ \in C^2(\Omega)$ *to problem* (P_+) *satisfies*

$$\lim_{d(x)\to 0} \frac{u_+(x)}{\varphi_1(d(x))} = 1, \tag{1.21}$$

where $\varphi_1 \in RVZ_{-q/(\rho+1-q)}$ *and there exists* $H \in RVZ_0$ *such that*

$$\varphi_1\big(d(x)\big) = H\big(d(x)\big)\big(d(x)\big)^{-q/(\rho+1-q)}. \tag{1.22}$$

Corollary 1.2. *If* $g(u) = u^{1+\rho}$, $\rho > 0$, *then*

(i) *if* $\frac{2(\rho+1)}{\rho+2} = q$, *let* $\zeta_0^\pm = (\gamma_0)^{2/q}(\xi_0^\pm)^{-\rho}$, *then any solution* $u_\pm$ *to problem* $(\mathrm{P}_\pm)$ *satisfies* (1.9);
(ii) *if* $\frac{2(1+\rho)}{\rho+2} < q < \rho+1$, *then* $\varphi_1(t) = (\frac{\rho+1-q}{q})^{q/(\rho+1-q)} t^{-q/(\rho+1-q)}$, $t > 0$, *any solution* u_+ *to problem* (P_+) *satisfies*

$$\lim_{d(x)\to 0} u_+(x)\big(d(x)\big)^{q/(\rho+1-q)} = \left(\frac{\rho+1-q}{q}\right)^{q/(\rho+1-q)}.$$

Theorem 1.3. *Let* g *satisfies* (g_1), $g' \in RV_\rho$ *with* $\rho > 0$, b *satisfies* (b_1) *and* (b_2) *with* $l_1 > 0$.

If $0 \leqslant q < 2(1+l_1\rho)/(2+l_1\rho)$, *then every solution* $u_\pm \in C^2(\Omega)$ *to problem* $(\mathrm{P}_\pm)$ *satisfies* (1.4), *where* $\xi_0 = (\frac{2+l_1\rho}{c_0(2+\rho)})^{1/\rho}$ *and* $\psi \in C^2(0,a]$ *is uniquely determined by* (1.5). *Moreover,* $\psi \in RVZ_{-2/l_1\rho}$, *and there exists* $H \in RVZ_0$ *such that*

$$\lim_{d(x)\to 0} \frac{u_\pm(x)}{H(d(x))(d(x))^{-2/l_1\rho}} = \xi_0. \tag{1.23}$$

Corollary 1.3. *If* $g(u) = u^{\rho+1}$, $\rho > 0$, *and* $b(x) \cong c_0(d(x))^\tau$ *with* $\tau \geqslant 0$ *near* $\partial\Omega$, $0 \leqslant q < (2+\tau+2\rho)/(2+\tau+\rho)$, *then*

$$k(t) = t^{\tau/2}, \qquad \psi(s) = cs^{-(2+\tau)/\rho}, \qquad c = \left(\frac{(\rho+2)(2+\tau)^2}{2\rho^2}\right)^{1/\rho},$$

$$\tau = \frac{2(1-l_1)}{l_1}, \qquad \xi_0^\rho = \frac{2(2+\rho+\tau)}{c_0(2+\tau)(2+\rho)},$$

and every solution $u_{\pm} \in C^2(\Omega)$ *to problem* $(\mathrm{P}_{\pm})$ *satisfies*

$$\lim_{d(x)\to 0} u_{\pm}(x)\big(d(x)\big)^{(2+\tau)/\rho} = \left(\frac{(2+\tau)(2+\tau+\rho)}{c_0\rho^2}\right)^{1/\rho}.$$

Remark 1.1. By (1.5), we see that the asymptotical behaviour (1.4) of $u_{\pm}$ is independent on $\pm|\nabla u_{\pm}|^q$ when $0 \leqslant q < 2(1+l_1\rho)/(2+l_1\rho)$.

Remark 1.2. $\xi_0^+ > 1$ in Theorem 1.2.

Remark 1.3. $\varphi(d(x)) = \varphi_1(\int_0^{d(x)} k(s)\,ds)$.

Remark 1.4. Let g satisfies (g_1), $g'(t) = u^{\rho}L(t)$, $\rho > 0$, L be slowly varying at infinity. By the following Lemma 2.2 and Corollary 2.2, we see that

(i) if $q = \frac{2(\rho+1)}{\rho+2}$ and $\lim_{s\to\infty} L(s) = (1+\rho)\gamma_0 \in (0,\infty)$, then $\lim_{u\to\infty} \frac{g(u)}{u^{q/(2-q)}} = \gamma_0$;
(ii) if $q > \frac{2(\rho+1)}{\rho+2}$, then $\lim_{u\to\infty} \frac{g(u)}{u^{q/(2-q)}} = 0$.

Remark 1.5. Some basic examples of g which satisfies (g_1) and $g' \in RV_{\rho}$ with $\rho > 0$, are*:*

(i) $g(u) = u^{\rho+1}$;
(ii) $g(u) = u^{\rho+1}(\ln(u+1))^{\beta}$, $\beta > 0$;
(iii) $g(u) = u^{\rho+1}\arctan u$;
(iv) $g(u) = c_0u^{\rho+1}\exp(\int_0^u \frac{y(s)}{s}\,ds)$, $u \geqslant 0$, where $c_0 > 0$, $y \in C[0,\infty)$ is nonnegative such that $\lim_{s\to 0^+} y(s)/s \in [0,\infty)$ and $\lim_{s\to\infty} y(s) = 0$.

Remark 1.6. [6, Remark 2.2] When $l_1 > 0$, $k \in RVZ_{(1-l_1)/l_1}$.

Some basic examples of $k \in K$ with $l_1 > 0$ are:

(i) $k(t) = t^{\tau/2}$, $\tau > 0$, where $l_1 = 2/(2+\tau)$.
(ii) $k(t) = -\tau/\ln t$, $\tau > 0$, where $l_1 = 1$.
(iii) $k(t) = t^{\tau}/\ln(1+t^{-1})$, $\tau > 0$, where $l_1 = 1/(1+\tau)$.
(iv) $k(t) = c_0t^{\tau/2}\exp(\int_t^a \frac{y(s)}{s}\,ds)$, $0 < t < a$, where $\tau > 0$, $c_0 > 0$, $y \in C[0,a]$ and $\lim_{t\to 0^+} y(t) = 0$, and $l_1 = 2/(2+\tau)$.

Remark 1.7. When $q = 2$, the change of variable $v = e^u - 1$ transforms problem (P_+) into the equivalent one

$$\Delta v = b(x)h(v), \quad v > 0,\ x \in \Omega,\ v|_{\partial\Omega} = \infty, \tag{1.24}$$

where $h(s) = (s+1)g(\ln(s+1))$.

Although problem (1.24) is the same type as problem (P_0), $h \in RV_1$, provided $g \in RV_{\rho+1}$ with $\rho > 0$ (see the following Lemma 2.3). Thus we cannot use the above results for problem (P_0).

The paper is organized as follows. In Section 2 we continue to recall Karamata regular variation theory. In Section 3 we prove Theorems 1.1–1.3. Finally, we show existence of solutions to problem $(P_\pm)$.

2. Some basic definitions and the properties to Karamata regular variation theory

Let us continue to recall some basic definitions and the properties to Karamata regular variation theory (see [33,36,37]).

Some basic examples of slowly varying functions at infinity are:

(i) every measure function on $[a, \infty)$ which has a positive limit at infinity.

(ii) $L(t) = \prod_{m=1}^{m=n} (\log_m t)^{\alpha_m}$, $\alpha_m \in \mathbb{R}$.

(iii) $L(t) = e^{(\prod_{m=1}^{m=n} (\log_m t)^{\alpha_m})}$, $0 < \alpha_m < 1$.

(iv) $L(t) = \frac{1}{t} \int_a^t \frac{ds}{\ln s}$.

(v) $L(t) = e^{((\ln t)^{1/3} \cos(\ln t)^{1/3})}$, where $\lim_{t\to\infty} \inf L(t) = 0$, $\lim_{t\to\infty} \sup L(t) = +\infty$.

Lemma 2.1. *If the functions L, L_1 are slowly varying at infinity, then*

(i) L^σ *for every* $\sigma \in \mathbb{R}$, $L(t) + L_1(t)$, $L(L_1(t))$ *(if* $L_1(t) \to \infty$ *as* $t \to \infty$*), are also slowly varying at infinity*;

(ii) *for every* $\theta > 0$ *and* $t \to \infty$,

$$t^\theta L(t) \to \infty, \qquad t^{-\theta} L(t) \to 0; \tag{2.1}$$

(iii) *for* $t \to \infty$, $\ln(L(t))/\ln t \to 0$.

Definition 2.1. A positive measure function H defined on some neighborhood $(0, a)$ for some $a > 0$, is called slowly varying at zero, if for each $\xi > 0$

$$\lim_{t\to 0^+} \frac{H(\xi t)}{H(t)} = 1. \tag{2.2}$$

It follows by Definitions 1.3 and 2.1 that if $h \in RVZ_\sigma$ then it can be represented in the form

$$h(t) = t^\sigma H(t). \tag{2.3}$$

Remark 2.1. Definition 1.1 is equivalent to saying that $f(1/t)$ is regularly varying at zero of index $-\rho$.

Lemma 2.2 *(Asymptotical behaviour). If the functions L is slowly varying at infinity, then for $t \to \infty$,*

$$\int_a^t s^\beta L(s)\, ds \cong (\beta + 1)^{-1} t^{1+\beta} L(t), \quad \text{for } \beta > -1. \tag{2.4}$$

Let Ψ be nondecreasing on $\mathbb{R}$, we define (as in [36]) the inverse of Ψ by

$$\Psi^{\leftarrow}(t) = \inf\{s\colon\ \Psi(s) \geqslant t\}. \tag{2.5}$$

Lemma 2.3. [36, Proposition 0.8] *The following hold*:

(i) *if* $f_1 \in RV_{\rho_1}$, $f_2 \in RV_{\rho_2}$, *then* $f_1 \cdot f_2 \in RV_{\rho_1+\rho_2}$;
(ii) *if* $f_1 \in RV_{\rho_1}$, $f_2 \in RV_{\rho_2}$, *with* $\lim_{t\to\infty} f_2(t) = \infty$, *then* $f_1 \circ f_2 \in RV_{\rho_1\rho_2}$;
(iii) *if* Ψ *is nondecreasing on* $\mathbb{R}$, $\lim_{t\to\infty} \Psi(t) = \infty$, *and* $\Psi \in RV_\rho$ *with* $\rho \neq 0$, *then* $\Psi^{\leftarrow} \in RV_{\rho^{-1}}$.

By the above lemmas, we can directly obtain the following results.

Corollary 2.1 *(Representation theorem). The function* H *is slowly varying at zero if and only if it may be written in the form*

$$H(t) = c(t)\exp\left(\int_t^a \frac{y(s)}{s}\,ds\right), \quad 0 < t < a, \tag{2.6}$$

for some $a > 0$, *where* $c(t)$ *is a bounded measurable function,* $y \in C[0,a]$ *and for* $t \to 0^+$, $y(t) \to 0$ *and* $c(t) \to c$, *with* $c > 0$.

Corollary 2.2. *If the function* H *is slowly varying at zero, then for every* $\theta > 0$ *and* $t \to 0^+$,

$$t^{-\theta} H(t) \to \infty, \qquad t^{\theta} H(t) \to 0. \tag{2.7}$$

Corollary 2.3. *If* g *satisfies* (g_1) *and* $g' \in RV_\rho$ *with* $\rho > 0$, *then*

(i) $g \in RV_{\rho+1}$ *and* $G \in RV_{\rho+2}$;
(ii) g *satisfies Keller–Osserman condition* (g_2).

Proof. (i) By Definition 1.1, the l'Hôpital rule and Lemma 2.2, we can show (i).

(ii) By $G \in RV_{\rho+2}$, we see that there exists a function L which is slowly varying at infinity such that $G(t) = t^{2+\rho}L(t)$. Since $\rho > 0$, let $\rho_1 \in (0, \rho/2)$, we see by Lemma 2.1 that

$$\lim_{t\to\infty} G(t)/t^{2(1+\rho_1)} = \lim_{t\to\infty} t^{\rho-2\rho_1} L(t) = \infty,$$

i.e., there exists $T_0 > 0$ such that

$$G(t)/t^{2(1+\rho_1)} > 1, \qquad \sqrt{G(t)} > t^{1+\rho_1}, \quad t > T_0.$$

This implies that g satisfies (g_2). □

Corollary 2.4. *If* $h_1 \in RVZ_{\rho_1}$, $h_2 \in RVZ_{\rho_2}$, *with* $\lim_{t\to 0^+} h_2(t) = 0$, *then* $h_1 \circ h_2 \in RVZ_{\rho_1\rho_2}$.

Lemma 2.4. *Let g_1 and g_2 be positive continuous on $(0,\infty)$, $g_1 \in RV_{1+\rho}$ with $\rho > 0$, $\int_t^\infty \frac{ds}{g_1(s)} < \infty$, $\forall t > 0$, and k be positive continuous and nondecreasing on $(0,a)$. If $\lim_{t\to\infty} \frac{g_1(t)}{g_2(t)} = 1$ and*

$$\int_{\varphi_1(t)}^{\infty} \frac{ds}{g_1(s)} = \int_0^t k(s)\,ds = \int_{\varphi_2(t)}^{\infty} \frac{ds}{g_2(s)}, \quad \forall t \in (0,a),$$

then $\lim_{t\to\infty} \frac{\varphi_1(t)}{\varphi_2(t)} = 1$.

Proof. First we note that $g_2 \in RV_{1+\rho}$ and for any given small $\varepsilon > 0$, there exists $t_1 > 0$ such that

$$g_1\big((1+\varepsilon)z\big) > (1+\varepsilon)^{\rho/2+1} g_1(z), \qquad g_2(z) < (1+\varepsilon)^{\rho/2} g_1(z), \quad \forall z > \phi_2(t),\ \forall t \in (0,t_1).$$

It follows that, for $t \in (0,t_1)$,

$$\int_{\varphi_1(t)}^{\infty} \frac{ds}{g_1(s)} = \int_{\varphi_2(t)}^{\infty} \frac{ds}{g_2(s)} > \int_{\varphi_2(t)}^{\infty} \frac{ds}{(1+\varepsilon)^{\rho/2} g_1(s)} > \int_{\varphi_2(t)}^{\infty} \frac{(1+\varepsilon)\,ds}{g_1((1+\varepsilon)s)} = \int_{(1+\varepsilon)\varphi_2(t)}^{\infty} \frac{dz}{g_1(z).}$$

This implies that

$$\varphi_1(t) < (1+\varepsilon)\varphi_2(t), \quad t \in (0,t_1).$$

Similarly we can show that there exists $t_2 > 0$ such that

$$\varphi_2(t) < (1+\varepsilon)\varphi_1(t), \quad t \in (0,t_2).$$

Therefore

$$\lim_{t\to\infty} \frac{\varphi_1(t)}{\varphi_2(t)} = 1. \qquad \square$$

Lemma 2.5. *Under the assumption in Theorem* 1.1, *$\varphi \in RVZ_{-q/l_1(\rho+1-q)}$. Moreover, if $q = \frac{2(\rho+1)}{\rho+2}$ and $\lim_{s\to\infty} L(s) = (1+\rho)\gamma_0 \in (0,\infty)$, then*

$$\lim_{t\to 0^+} \varphi(t)\left(\int_0^t k(s)\,ds\right)^{(2-q)/(q-1)} = \left(\frac{2-q}{\gamma_0^{1/q}(q-1)}\right)^{(2-q)/(q-1)}.$$

Proof. Let $f_1(u) = \int_u^\infty \frac{ds}{(g(s))^{1/q}}$, $u > 0$ and $f_2(t) = \int_0^t k(s)\,ds$, $\forall t \in (0,a)$. By the l'Hôpital rule and Remark 1.6, we can easily see that $f_1 \in RV_{-(\rho+1-q)/q}$ and $f_2 \in RVZ_{1/l_1}$. It follows by Corollary 2.4, Lemma 2.3 and Remark 2.1 that $\varphi_1 = f_1^{-1} \in RVZ_{-q/(\rho+1-q)}$ and $\varphi = f_1^{-1} \circ f_2 \in RVZ_{-q/l_1(\rho+1-q)}$. Moreover, if $q = \frac{2(\rho+1)}{\rho+2}$ and $\lim_{s\to\infty} L(s) = (1+\rho)\gamma_0 \in (0,\infty)$, then $\lim_{s\to\infty} \frac{g(s)}{\gamma_0 s^{q/(2-q)}} = 1$. The result follows by Lemma 2.4. $\square$

Lemma 2.6. *Under the assumption in Theorem* 1.3, $\psi \in RVZ_{-2/l_1\rho}$.

Proof. Let $f_{11}(u) = \int_u^\infty \frac{ds}{\sqrt{2G(s)}}$, and $f_2(t) = \int_0^t k(s)\,ds$, $\forall t \in (0, \tau)$. By the l'Hôpital rule, we can easily see that $f_{11} \in RV_{-\rho/2}$, $f_2 \in RVZ_{1/l_1}$. It follows by Corollary 2.4, Lemma 2.3 and Remark 2.1 that $\psi = f_{11}^{-1} \circ f_2 \in RVZ_{-2/l_1\rho}$. $\square$

3. The exact asymptotical behaviour

In this section we prove Theorems 1.1–1.3.

First we still need some preliminary considerations.

Lemma 3.1. *Assume g satisfies* (g_1) *and $g' \in RV_\rho$ with $\rho > 0$, then*

(i)

$$\lim_{s\to\infty} \frac{g'(s)}{g(s)} = 0, \qquad \lim_{s\to\infty} \frac{sg'(s)}{g(s)} = 1 + \rho;$$

(ii) *for* $0 < q < \rho + 1$,

$$\int_t^\infty \frac{ds}{(g(s))^{1/q}} < \infty, \quad \forall t > 0.$$

(iii)

$$\lim_{s\to\infty} \frac{g'(s)G(s)}{g^2(s)} = \frac{1+\rho}{2+\rho}.$$

Proof. (i) Since g satisfies (g_1) and $g' \in RV_\rho$ with $\rho > 0$, we see that there exists a function L which is slowly varying at infinity such that $g'(t) = t^\rho L(t)$. It follows by Lemma 2.2 that

$$g(t) = \int_0^t g'(s)\,ds = \int_0^t s^\rho L(s)\,ds \cong (\rho+1)^{-1} t^{1+\rho} L(t).$$

So

$$\lim_{s\to\infty} \frac{g'(s)}{g(s)} = 0, \qquad \lim_{s\to\infty} \frac{sg'(s)}{g(s)} = \rho + 1.$$

(ii) By $g^{1/q} \in RV_{(\rho+1)/q}$ with $(\rho+1)/q > 1$, we see that there exists a function L which is slowly varying at infinity such that $(g(t))^{1/q} = t^{(\rho+1)/q} L(t)$. It follows by Lemma 2.1 that $\lim_{t\to\infty} s^{(\rho+1)/q-\rho_1} L(t) = \infty$ for all $\rho_1 \in (1, (\rho+1)/q)$. Thus there exists $T_0 > 0$ such that $(g(t))^{1/q} = t^{\rho_1} t^{(\rho+1)/q-\rho_1} L(t) > t^{\rho_1}$, $\forall t > T_0$, i.e., $(g(t))^{1/q} > t^{\rho_1}$, $\forall t > T_0$. This implies that (ii) holds.

(iii) As the same way as the proof of (i), we can have

$$\lim_{s\to\infty}\frac{G(s)}{sg(s)}=(2+\rho)^{-1},$$

and

$$\lim_{s\to\infty}\frac{g'(s)G(s)}{g^2(s)}=\lim_{s\to\infty}\frac{sg'(s)}{g(s)}\lim_{s\to\infty}\frac{G(s)}{sg(s)}=\frac{1+\rho}{2+\rho}. \qquad \square$$

Lemma 3.2. *Let g and φ be as in Theorem* 1.1.

(i) $\lim_{t\to 0^+}\varphi(t)=\infty$, $\lim_{t\to 0^+}\varphi'(t)=-\infty$;
(ii) *if either $q=\frac{2(\rho+1)}{\rho+2}$ and $\lim_{s\to\infty}L(s)=(1+\rho)\gamma_0\in(0,\infty)$, or $\frac{2(\rho+1)}{\rho+2}<q\leqslant 2$, then*

$$\lim_{t\to 0^+}\frac{\varphi''(t)}{(-\varphi'(t))^q}=0.$$

Proof. First we note by Remark 1.2 that

$$\lim_{u\to\infty}\frac{g(u)}{u^{q/(2-q)}}=\hat{\gamma}_0=\begin{cases}0, & \text{for } q>\frac{2(\rho+1)}{\rho+2},\\ \gamma_0, & \text{for } q=\frac{2(\rho+1)}{\rho+2}.\end{cases}$$

By the definition of φ in (1.16) and a direct calculation, we see that

$$\varphi'(t)=-k(t)\big(g\big(\varphi(t)\big)\big)^{1/q}, \quad t\in(0,a);$$
$$\varphi''(t)=-k'(t)\big(g\big(\varphi(t)\big)\big)^{1/q}+\frac{(k(t))^2}{q}g'\big(\varphi(t)\big)\big(g\big(\varphi(t)\big)\big)^{(2-q)/q}, \quad t\in(0,a).$$

Since, for $q=\frac{2(\rho+1)}{\rho+2}$ and $\lim_{s\to\infty}L(s)=(1+\rho)\gamma_0\in(0,\infty)$, or $\frac{2(\rho+1)}{\rho+2}<q<2$,

$$\begin{aligned}&\lim_{t\to 0^+}\frac{(g(\varphi(t)))^{(1-q)/q}}{\int_0^t k(s)\,ds}\\ &\quad=\lim_{u\to\infty}\frac{(g(u))^{(1-q)/q}}{\int_u^\infty\frac{ds}{(g(u))^{1/q}}}=\frac{q-1}{q}\lim_{u\to\infty}g'(u)\big(g(u)\big)^{2(1-q)/q}\\ &\quad=\frac{q-1}{q}\lim_{u\to\infty}\frac{ug'(u)}{g(u)}\lim_{u\to\infty}\left(\frac{g(u)}{u^{q/(2-q)}}\right)^{(2-q)/q}=\frac{(q-1)(\rho+1)}{q}\hat{\gamma}_0^{(2-q)/q}\end{aligned}$$

and

$$\lim_{t\to 0^+}\frac{k'(t)}{(k(t))^2}\big(g\big(\varphi(t)\big)\big)^{(1-q)/q}=\lim_{t\to 0^+}\frac{k'(t)\int_0^t k(s)\,ds}{(k(t))^2}\lim_{t\to 0^+}\frac{(g(\varphi(t)))^{(1-q)/q}}{\int_0^t k(s)\,ds}$$

$$
\begin{aligned}
&= \frac{\rho+1}{q}\gamma_0^{(2-q)/q} \lim_{t\to 0^+}\left(1-\frac{d}{dt}\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)\right)\\
&= \frac{(1-l_1)(\rho+1)}{q}\hat{\gamma}_0{}^{(2-q)/q}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\lim_{t\to 0^+}\frac{\varphi''(t)}{(-\varphi'(t))^q} = &-\lim_{t\to 0^+}\big(k(t)\big)^{2-q}\lim_{t\to 0^+}\frac{k'(t)\int_0^t k(s)\,ds}{(k(t))^2}\lim_{t\to 0^+}\frac{(g(\varphi(t)))^{(1-q)/q}}{\int_0^t k(s)\,ds}\\
&+\frac{1}{q}\lim_{t\to 0^+}\big(k(t)\big)^{2-q}\lim_{t\to 0^+}\frac{g'(\varphi(t))(g(\varphi(t)))^{(2-q)/q}}{g(\varphi(t))}=0
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{t\to 0^+}\frac{\varphi'(t)}{(-\varphi'(t))^q} = &-\lim_{t\to 0^+}\big(k(t)\big)^{2-q}\lim_{t\to 0^+}\frac{\int_0^t k(s)\,ds}{k(t)}\lim_{t\to 0^+}\frac{(g(\varphi(t)))^{(1-q)/q}}{\int_0^t k(s)\,ds}\\
&+\frac{1}{q}\lim_{t\to 0^+}\big(k(t)\big)^{2-q}\lim_{u\to\infty} g'(u)\big(g(u)\big)^{2(1-q)/q}=0.
\end{aligned}
$$

For $q=2$,

$$
\begin{aligned}
\lim_{t\to 0^+}\frac{\varphi''(t)}{(-\varphi'(t))^2} &= -\lim_{t\to 0^+}\frac{k'(t)\int_0^t k(s)\,ds}{(k(t))^2}\lim_{t\to 0^+}\frac{(g(\varphi(t)))^{-1/2}}{\int_0^t k(s)\,ds}+\frac{1}{2}\lim_{u\to\infty}\frac{g'(u)}{g(u)}\\
&= -\lim_{t\to 0^+}\frac{k'(t)\int_0^t k(s)\,ds}{(k(t))^2}\lim_{u\to\infty}\frac{(g(u))^{-1/2}}{\int_u^\infty \frac{ds}{\sqrt{g(s)}}}\\
&= -\frac{1}{2}\lim_{t\to 0^+}\frac{k'(t)\int_0^t k(s)\,ds}{(k(t))^2}\lim_{u\to\infty}\frac{1}{g(u)}=0
\end{aligned}
$$

and

$$
\lim_{t\to 0^+}\frac{\varphi'(t)}{(-\varphi'(t))^2} = -\lim_{t\to 0^+}\frac{\int_0^t k(s)\,ds}{k(t)}\lim_{t\to 0^+}\frac{(g(\varphi(t)))^{1/2}}{\int_0^t k(s)\,ds}=0. \qquad \square
$$

As the same proof as Lemma 3.2, we can show:

Lemma 3.3. *Let g and φ_1 be as in Theorem* 1.2.

(i) $\lim_{t\to 0^+}\varphi_1(t)=\infty$, $\lim_{t\to 0^+}\varphi_1'(t)=-\infty$;
(ii) *if* $q=\frac{2(\rho+1)}{\rho+2}$ *and* $\lim_{s\to\infty}L(s)=(1+\rho)\gamma_0\in(0,\infty)$, *then*

$$
\lim_{t\to 0^+}\frac{\varphi_1''(t)}{(-\varphi_1'(t))^q}=\frac{\rho+1}{q}\gamma_0^{(2-q)/q};
$$

(iii) *if* $\frac{2(\rho+1)}{\rho+2} < q$*, then*

$$\lim_{t\to 0^+} \frac{\varphi_1''(t)}{(-\varphi_1'(t))^q} = 0.$$

Lemma 3.4. *Let* g*,* k *and* ψ *be as in Theorem* 1.3*, then*

(i) $\lim_{t\to 0^+} \frac{k'(t)\sqrt{2G(\psi(t))}}{k^2(t)g(\psi(t))} = \frac{\rho(1-l_1)}{\rho+2}$;

(ii) $\lim_{t\to 0^+} \frac{\sqrt{2G(\psi(t))}}{k(t)g(\psi(t))} = 0$;

(iii) *if* $q < 2(1+l_1\rho)/(2+l_1\rho)$*, then* $\lim_{t\to 0^+} \frac{(k(t)\sqrt{2G(\psi(t))})^q}{k^2(t)g(\psi(t))} = 0.$

Proof. By (1.5), we see by a direct calculation that

$$\psi'(t) = -k(t)\sqrt{2G\big(\psi(t)\big)}, \qquad \psi''(t) = k'(t)\sqrt{2G\big(\psi(t)\big)} + k^2(t)g\big(\psi(t)\big), \quad 0 < t < a.$$

(i) By (b_2) and Lemma 3.1, we see that

$$\begin{aligned}
\lim_{t\to 0^+} \frac{k'(t)\sqrt{2G(\psi(t))}}{k^2(t)g(\psi(t))} &= \lim_{t\to 0^+} \frac{k'(t)\int_0^t k(s)\,ds}{k^2(t)} \lim_{t\to 0^+} \frac{\sqrt{2G(\psi(t))}/g(\psi(t))}{\int_0^t k(s)\,ds} \\
&= \lim_{t\to 0^+}\left(1 - \frac{d}{dt}\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)\right) \lim_{u\to\infty} \frac{\sqrt{2G(u)}/g(u)}{\int_u^\infty \frac{ds}{\sqrt{2G(s)}}} \\
&= \frac{\sigma}{(2+\sigma)}\left(-1 + 2\lim_{u\to\infty} \frac{g'(u)G(u)}{(g(u))^2}\right) \\
&= \frac{\sigma}{(2+\sigma)}\left(-1 + 2\frac{1+\rho}{2+\rho}\right) \\
&= \frac{\sigma\rho}{(2+\sigma)(2+\rho)}.
\end{aligned}$$

(ii) By (b_2) and the proof of (i), we see that

$$\begin{aligned}
\lim_{t\to 0^+} \frac{\sqrt{2G(\psi(t))}}{k(t)g(\psi(t))} &= \lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{k(t)} \lim_{t\to 0^+} \frac{\sqrt{2G(\psi(t))}/g(\psi(t))}{\int_0^t k(s)\,ds} \\
&= \lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{k(t)} \lim_{u\to\infty} \frac{\sqrt{2G(u)}/g(u)}{\int_u^\infty \frac{ds}{\sqrt{2G(s)}}} = 0.
\end{aligned}$$

(iii) By Corollaries 2.3 and 2.4, Lemmas 2.3 and 2.6, we see that

$$g\circ\psi \in RVZ_{-2(\rho+1)/l_1\rho}; \qquad \frac{1}{g\circ\psi} \in RVZ_{2(\rho+1)/l_1\rho};$$
$$k^{q-2} \in RVZ_{(q-2)(1-l_1)/l_1}; \qquad G\circ\psi \in RVZ_{-2(2+\rho)/l_1\rho};$$
$$(G\circ\psi)^{q/2} \in RVZ_{-q(2+\rho)/l_1\rho}.$$

Thus

$$k^{q-2} \cdot (G \circ \psi)^{q/2} \cdot \frac{1}{g \circ \psi} \in RVZ_{\rho_0},$$

where $\rho_0 = \frac{1}{l_1 \rho}[(2 + 2l_1\rho - q(2 + l_1\rho)] > 0$.

The proof is finished. □

Lemma 3.5. *If g satisfies* (g$_1$)*, $g' \in RV_\rho$ with $\rho > 0$, b satisfies* (b$_1$)*, then:*

(i) *problem* (P$_+$) *has at least one solution $u_+ \in C^2(\Omega)$ for $q \geqslant 0$;*
(ii) *the same statement is true to problem* (P$_-$) *for $q \in [0, 2]$; and*
(iii) *the same statement is true to the following problem*

$$\Delta w \pm |\nabla w|^q = b(x)g(w), \quad x \in \Omega / \overline{\Omega}_0,\ w|_{\partial\Omega} = 1,\ w|_{\partial\Omega_0} = +\infty, \tag{Q$_\pm$}$$

where a solution $w_\pm \in C^2(\overline{\Omega}/\overline{\Omega}_0)$, $\Omega_0 \Subset \Omega$ and $\partial\Omega_0$ is a smooth submanifold of dimension $N - 1$.

In Appendix A, we will give the proof.

Proof of Theorem 1.1. First we note that $\xi_0^+ = c_q^{-1/(\rho-q+1)}$ is the unique positive solution of

$$\xi^{q-1} - c_q \xi^\rho = 0.$$

Since $\rho > q - 1 > 0$, for any $\varepsilon \in (0, \mu)$ with $\mu = \frac{c_q (\xi_0^+)^\rho [c_0^+(\rho+1-q)-2\rho]}{4\rho c_0^+(c_0^+-2)}$, we can define

$$\xi_{2\varepsilon}^+ = \left(\frac{c_q(\xi_0^+)^\rho + c_0^+\varepsilon}{c_q}\right)^{1/\rho}, \qquad \xi_{1\varepsilon}^+ = \left(\frac{c_q(\xi_0^+)^\rho - c_0^+\varepsilon}{c_q}\right)^{1/\rho},$$

where $c_0^+ > \frac{2\rho}{\rho+1-q}$, i.e., $c_0^+ > 2$ satisfies

$$\frac{c_0^+ - 2}{c_0^+} > \frac{q-1}{\rho}.$$

For $t \in (0, 1)$ and $a \in (0, 1)$, by the basic inequalities

$$(1 + t)^a - 1 < at \quad \text{and} \quad 1 - (1 - t)^a < \frac{at}{1 - t}; \tag{3.1}$$

we see that

$$\begin{aligned}
\left(\xi_{2\varepsilon}^+\right)^{q-1} - \left(\xi_0^+\right)^{q-1} &= \left(\xi_0^+\right)^{q-1}\left[\left(1 + \frac{c_0^+\varepsilon}{c_q(\xi_0^+)^\rho}\right)^{(q-1)/\rho} - 1\right] < \left(\xi_0^+\right)^{q-1}\frac{q-1}{\rho}\frac{c_0^+\varepsilon}{c_q(\xi_0^+)^\rho} \\
&= \frac{c_0^+\varepsilon(q-1)}{\rho} < \left(c_0^+ - 2\right)\varepsilon
\end{aligned}$$

and

$$\big(\xi_0^+\big)^{q-1} - \big(\xi_{1\varepsilon}^+\big)^{q-1} = \big(\xi_0^+\big)^{q-1}\bigg[1 - \bigg(1 - \frac{c_0^+\varepsilon}{c_q(\xi_0^+)^\rho}\bigg)^{(q-1)/\rho}\bigg]$$
$$< \frac{c_0^+\varepsilon(q-1)}{\rho}\frac{c_q(\xi_0^+)^\rho}{c_q(\xi_0^+)^\rho - c_0^+\varepsilon} = \frac{c_0^+\varepsilon(q-1)}{\rho}\frac{\rho(c_0^+-2)}{c_0^+(q-1)} < \big(c_0^+-2\big)\varepsilon.$$

For any $\delta > 0$, we define $\Omega_\delta = \{x \in \Omega\colon\ d(x) < \delta\}$, and $\partial\Omega_\delta = \{x \in \Omega\colon\ d(x) = \delta\}$. By Lemma 3.2 and the regularity of $\partial\Omega$, we can choose δ sufficiently small such that

(i) $d(x) \in C^2(\overline{\Omega}_{2\delta})$;
(ii) $|(\frac{\varphi''(s)}{(-\varphi'(s))^q} + \frac{\varphi'(s)}{(-\varphi'((s))^q}\Delta d(x)| < \varepsilon$, for all $(x,s) \in \Omega_{2\delta} \times (0,2\delta)$, $i = 1,2$;
(iii) $|\frac{b(x)g(\xi_{i\varepsilon}^+(\varphi(s)))}{k^q(d(x))\xi_{i\varepsilon}^+ g(\varphi(s))} - c_q(\xi_{i\varepsilon}^+)^\rho| < \varepsilon$, for all $(x,s) \in \Omega_{2\delta} \times (0,2\delta)$, $i = 1,2$.

Let $\beta \in (0,\delta)$ be arbitrary. We define $\bar u_\beta = \xi_{2\varepsilon}^+\varphi(d(x)-\beta)$ for any x with $\beta < d(x) < 2\delta$, and $\underline{u}_\beta = \xi_{1\varepsilon}^+\varphi(d(x)+\beta)$, for any x with $d(x)+\beta < 2\delta$. It follows by $|\nabla d(x)| = 1$ that

$$\begin{aligned}
&\Delta\bar u_\beta(x) - b(x)g\big(\bar u_\beta(x)\big) + \big|\nabla\bar u_\beta(x)\big|^q\\
&\quad= \xi_{2\varepsilon}^+\varphi''\big(d(x)-\beta\big) + \xi_{2\varepsilon}^+\varphi'\big(d(x)-\beta\big)\Delta d(x)\\
&\qquad+ \big(\xi_{2\varepsilon}^+\big)^q\big(-\varphi'\big(d(x)-\beta\big)\big)^q - b(x)g\big(\xi_{2\varepsilon}^+\varphi\big(d(x)-\beta\big)\big)\\
&\quad= \xi_{2\varepsilon}^+\big(k\big(d(x)-\beta\big)\big)^q g\big(\varphi\big(d(x)-\beta\big)\big)\bigg[\frac{\varphi''(d(x)-\beta)}{(-\varphi'(d(x)-\beta)^q}\\
&\qquad- \bigg(\frac{b(x)g(\xi_{2\varepsilon}^+(\varphi(d(x)-\beta)))}{\xi_{2\varepsilon}^+(k(d(x)-\beta))^q g(\varphi(d(x)-\beta))} - c_q\big(\xi_{2\varepsilon}^+\big)^\rho\bigg)\\
&\qquad- c_q\big(\xi_{2\varepsilon}^+\big)^\rho + \big(\xi_{2\varepsilon}^+\big)^{q-1} + \frac{\varphi'(d(x)-\beta)}{(-\varphi'(d(x)-\beta))^q}\Delta d(x)\bigg]\\
&\quad\leqslant \xi_{2\varepsilon}^+ g\big(\varphi\big(d(x)-\beta\big)\big)\big[2\varepsilon - c_0^+\varepsilon + \big(\xi_{2\varepsilon}^+\big)^{q-1} - \big(\xi_0^+\big)^{q-1}\big] \leqslant 0
\end{aligned}$$

and

$$\begin{aligned}
&\Delta\underline{u}_\beta(x) - b(x)g\big(\underline{u}_\beta(x)\big) + \big|\nabla\underline{u}_\beta(x)\big|^q\\
&\quad= \xi_{1\varepsilon}^+\varphi''\big(d(x)+\beta\big) + \xi_{1\varepsilon}^+\varphi'\big(d(x)+\beta\big)\Delta d(x)\\
&\qquad+ \big(\xi_{1\varepsilon}^+\big)^q\big(-\varphi'\big(d(x)+\beta\big)\big)^q - b(x)g\big(\xi_{1\varepsilon}^+\varphi\big(d(x)+\beta\big)\big)\\
&\quad= \xi_{1\varepsilon}^+\big(k\big(d(x)+\beta\big)\big)^q g\big(\varphi\big(d(x)+\beta\big)\big)\bigg[\frac{\varphi''(d(x)+\beta)}{(-\varphi'(d(x)+\beta))^q}\\
&\qquad- \bigg(\frac{b(x)g(\xi_{1\varepsilon}^+(\varphi(d(x)+\beta)))}{\xi_{1\varepsilon}^+(k(d(x)+\beta))^q g(\varphi(d(x)+\beta))} - c_q\big(\xi_{1\varepsilon}^+\big)^\rho\bigg)
\end{aligned}$$

$$- c_q\big(\xi_{1\varepsilon}^+\big)^\rho + \big(\xi_{1\varepsilon}^+\big)^{q-1} + \frac{\varphi'(d(x)+\beta)}{(-\varphi'(d(x)+\beta))^q}\Delta d(x)\Big]$$

$$\geqslant \xi_{1\varepsilon}^+ g\big(\varphi\big(d(x)+\beta\big)\big)\big[-2\varepsilon + c_0^+\varepsilon + \big(\xi_{1\varepsilon}^+\big)^{q-1} - \big(\xi_0^+\big)^{q-1}\big] \geqslant 0.$$

Now let w_+ be an arbitrary solution of problem (Q_+) with $\Omega_0 = \Omega_\delta$ and u_+ be an arbitrary solution of problem (P_+) and $v_+ = u_+ + w_+$. We see that

$$u_+ + w_+|_{\partial\Omega} = +\infty > \underline{u}_\beta|_{\partial\Omega}, \qquad u_+ + w_+|_{\partial\Omega_\delta} = +\infty > \underline{u}_\beta|_{\partial\Omega_\delta},$$
$$\bar{u}_\beta + w_+|_{\partial\Omega_\beta} = +\infty > u|_{\partial\Omega_\beta}, \qquad \bar{u}_\beta + w_+|_{\partial\Omega_\delta} = +\infty > u_+|_{\partial\Omega_\delta}.$$

It follows by (g_1) and the following Lemma A.1 that

$$\underline{u}_\beta(x) \leqslant u_+(x) + w_+(x), \quad \forall x \in \Omega_\delta, \qquad u_+(x) \leqslant \bar{u}_\beta(x) + w_+(x), \quad \forall x \in \Omega_\delta \cap \Omega_\beta.$$

Let $\beta \to 0$, we see that

$$\xi_{1\varepsilon}^+\varphi\big(d(x)\big) \leqslant u_+(x) + w_+(x) \leqslant \xi_{2\varepsilon}^+\varphi\big(d(x)\big) + 2w_+(x), \quad \forall x \in \Omega_\delta,$$

which implies

$$\xi_{1\varepsilon}^+ \leqslant \liminf_{d(x)\to 0} \frac{u_+(x)}{\varphi(d(x))} \leqslant \limsup_{d(x)\to 0} \frac{u_+(x)}{\varphi(d(x))} \leqslant \xi_{2\varepsilon}^+.$$

Let $\varepsilon \to 0$, and look at the definitions of $\xi_{1\varepsilon}^+$ and $\xi_{2\varepsilon}^+$, we have

$$\lim_{d(x)\to 0} \frac{u_+(x)}{\varphi(d(x))} = \xi_0^+.$$

By Lemma 2.5, the proof is finished. □

Proof of Theorem 1.2. Since $\xi_0^+ > 1$ and $\rho > q-1 > 0$, for

$$0 < \varepsilon < \min\left\{\frac{(\xi_0^+)^\rho - (\xi_0^+)^{q-1}}{2c_0}, \frac{(\xi_0^-)^\rho}{2}\right\},$$

we can define

$$\xi_{2\varepsilon}^+ = \big(\big(\xi_0^+\big)^\rho + c_0^+\varepsilon\big)^{1/\rho}, \qquad \xi_{1\varepsilon}^+ = \big(\big(\xi_0^+\big)^\rho - c_0^+\varepsilon\big)^{1/\rho},$$
$$\xi_{2\varepsilon}^- = \big(\big(\xi_0^-\big)^\rho + c_0^-\varepsilon\big)^{1/\rho}, \qquad \xi_{1\varepsilon}^- = \big(\big(\xi_0^-\big)^\rho - c_0^-\varepsilon\big)^{1/\rho},$$

where $c_0^- = 2$ and $c_0^+ > 2$ satisfies

$$\frac{c_0^+ - 2}{c_0^+} > \frac{q-1}{\rho}.$$

By the basic inequalities (3.1), we see that

$$\begin{aligned}\big(\xi_{2\varepsilon}^+\big)^{q-1}-\big(\xi_0^+\big)^{q-1}&=\big(\xi_0^+\big)^{q-1}\left[\left(1+\frac{c_0^+\varepsilon}{(\xi_0^+)^\rho}\right)^{(q-1)/\rho}-1\right]\\&<\frac{q-1}{\rho}\frac{c_0^+\varepsilon}{(\xi_0^+)^{\rho-q+1}}<\frac{c_0^+\varepsilon(q-1)}{\rho}<\big(c_0^+-2\big)\varepsilon\end{aligned}$$

and

$$\begin{aligned}\big(\xi_0^+\big)^{q-1}-\big(\xi_{1\varepsilon}^+\big)^{q-1}&=\big(\xi_0^+\big)^{q-1}\left[1-\left(1-\frac{c_0^+\varepsilon}{(\xi_0^+)^\rho}\right)^{(q-1)/\rho}\right]\\&<\frac{q-1}{\rho}\frac{c_0^+\varepsilon}{(\xi_0^+)^\rho}\frac{(\xi_0^+)^{q-1}}{1-\frac{c_0^+\varepsilon}{(\xi_0^+)^\rho}}=\frac{c_0^+\varepsilon(q-1)}{\rho}\frac{(\xi_0^+)^{q-1}}{(\xi_0^+)^\rho-c_0^+\varepsilon}\\&<\frac{c_0^+\varepsilon(q-1)}{\rho}\frac{2(\xi_0^+)^{q-1}}{(\xi_0^+)^\rho+(\xi_0^+)^{q-1}}<\frac{c_0^+\varepsilon(q-1)}{\rho}<\big(c_0^+-2\big)\varepsilon.\end{aligned}$$

Obviously,

$$\big(\xi_0^-\big)^{q-1}-\big(\xi_{2\varepsilon}^-\big)^{q-1}<0\quad\text{and}\quad\big(\xi_0^-\big)^{q-1}-\big(\xi_{1\varepsilon}^-\big)^{q-1}>0.$$

By Lemma 3.3 and the regularity of $\partial\Omega$, we can choose δ sufficiently small such that

(i) $d(x)\in C^2\big(\overline{\Omega}_{2\delta}\big)$,

(ii) $\displaystyle\left|\left(\frac{\varphi_1''(s)}{(-\varphi_1'(s))^q}-\frac{\rho+1}{q}\gamma_0^{(2-q)/q}\right)+\frac{\varphi_1'(s)}{(-\varphi_1'((s))^q}\Delta d(x)\right|<\varepsilon,$

for all $(x,s)\in\Omega_{2\delta}\times(0,2\delta)$, $i=1,2$,

(iii) $\displaystyle\left|\frac{g(\xi_{i\varepsilon}^\pm(\varphi_1(s)))}{\xi_{i\varepsilon}^\pm g(\varphi_1(s))}-(\xi_{i\varepsilon}^\pm)^\rho\right|<\varepsilon,$ for all $s\in(0,2\delta)$, $i=1,2$.

Let $\beta\in(0,\delta)$ be arbitrary. We define $\bar u_\beta=\xi_{2\varepsilon}^\pm\varphi_1(d(x)-\beta)$ for any x with $\beta<d(x)<2\delta$, and $\underline{u}_\beta=\xi_{1\varepsilon}^\pm\varphi_1(d(x)+\beta)$, for any x with $d(x)+\beta<2\delta$. It follows by $|\nabla d(x)|=1$ that

$$\begin{aligned}&\Delta\bar u_\beta(x)-g\big(\bar u_\beta(x)\big)\pm\big|\nabla\bar u_\beta(x)\big|^q\\&=\xi_{2\varepsilon}^\pm g\big(\varphi_1\big(d(x)-\beta\big)\big)\left[\left(\frac{\varphi_1''(d(x)-\beta)}{(-\varphi_1'(d(x)-\beta))^q}-\frac{\rho+1}{q}\gamma_0^{(2-q)/q}\right)\right.\\&\quad-\left(\frac{g(\xi_{2\varepsilon}^\pm(\varphi_1(d(x)-\beta)))}{\xi_{2\varepsilon}^\pm g(\varphi_1(d(x)-\beta))}-\big(\xi_{2\varepsilon}^\pm\big)^\rho\right)\\&\quad\left.+\frac{\rho+1}{q}\gamma_0^{(2-q)/q}-\big(\xi_{2\varepsilon}^\pm\big)^\rho\pm\big(\xi_{2\varepsilon}^\pm\big)^{q-1}+\frac{\varphi_1'(d(x)-\beta)}{(-\varphi_1'((d(x)-\beta))^q}\Delta d(x)\right]\\&\leqslant\xi_{2\varepsilon}^\pm g\big(\varphi_1\big(d(x)-\beta\big)\big)\big[2\varepsilon-c_0^\pm\varepsilon\pm\big(\xi_{2\varepsilon}^\pm\big)^{q-1}\mp\big(\xi_0^\pm\big)^{q-1}\big]\leqslant 0\end{aligned}$$

and

$$
\begin{aligned}
&\Delta \underline{u}_\beta(x) - g\big(\underline{u}_\beta(x)\big) \pm \big|\nabla \underline{u}_\beta(x)\big|^q \\
&= \xi_{1\varepsilon}^{\pm} g\big(\varphi_1\big(d(x)+\beta\big)\big)\Bigg[\bigg(\frac{\varphi_1''((d(x)+\beta))}{(-\varphi_1'(d(x)+\beta))^q} - \frac{\rho+1}{q}\gamma_0^{(2-q)/q}\bigg) \\
&\quad - \bigg(\frac{g(\xi_{1\varepsilon}^{\pm}(\varphi_1(d(x)+\beta)))}{\xi_{1\varepsilon}^{\pm} g(\varphi_1(d(x)+\beta))} - \big(\xi_{1\varepsilon}^{\pm}\big)^\rho\bigg) \\
&\quad + \frac{\rho+1}{q}\gamma_0^{(2-q)/q} - \big(\xi_{1\varepsilon}^{\pm}\big)^\rho \pm \big(\xi_{1\varepsilon}^{\pm}\big)^{q-1} + \frac{\varphi_1'((d(x)+\beta)}{(-\varphi_1'(d(x)+\beta))^q}\Delta d(x)\Bigg] \\
&\geqslant \xi_{1\varepsilon}^{\pm} g\big(\varphi_1\big(d(x)+\beta\big)\big)\big[-2\varepsilon + c_0^{\pm}\varepsilon \pm \big(\xi_{1\varepsilon}^{\pm}\big)^{q-1} \mp \big(\xi_0^{\pm}\big)^{q-1}\big] \geqslant 0.
\end{aligned}
$$

As the same proof as that one of Theorem 1.1, we can obtain

$$
\lim_{d(x)\to 0} \frac{u_{\pm}(x)}{\varphi_1(d(x))} = \xi_0^{\pm}.
$$

By Lemma 2.5, the proof is finished. □

Proof of Theorem 1.3. Let

$$
\begin{aligned}
\tau_0 &= \frac{\sigma\rho}{(2+\sigma)(\rho+2)}, \qquad \xi_0 = \big((1-\tau_0)/c_0\big)^{1/\rho}, \\
\xi_{2\varepsilon} &= \big(\xi_0^\rho + 2\varepsilon/c_0\big)^{1/\rho}, \qquad \xi_{1\varepsilon} = \big(\xi_0^\rho - 2\varepsilon/c_0\big)^{1/\rho},
\end{aligned}
$$

where $\varepsilon \in (0, c_0\xi_0^\rho/4)$. One can easily see that

$$
\frac{\xi_0}{2^{1/\rho}} < \xi_{1\varepsilon} < \xi_0 < \xi_{2\varepsilon} < \left(\frac{3}{2}\right)^{1/\rho}\xi_0.
$$

By Lemma 3.4 and the regularity of $\partial\Omega$, we can choose δ sufficiently small such that

(i) $d(x) \in C^2\big(\overline{\Omega}_{2\delta}\big)$,

(ii) $$\left|\left(\frac{k'(s)\sqrt{2G(\psi(s))}}{k^2(s)g(\psi(s))} - \tau_0\right) + \frac{k(s)\sqrt{2G(\psi(s))}}{k^2(s)g(\psi(s))}\Delta d(x) + \frac{\xi_{i\varepsilon}^{q-1}k^q(s)(2G(\psi(s))^{q/2}}{k^2(s)g(\psi(s)))}\right| < \varepsilon,$$

for all $(x,s) \in \Omega_{2\delta} \times (0, 2\delta)$, $i = 1, 2$,

(iii) $$\frac{\xi_{2\varepsilon}k^2(d(x))g(\psi(d(x)))}{g(\xi_{2\varepsilon}\psi(d(x)))}\big(c_0\xi_{2\varepsilon}^\rho - \varepsilon\big) < b(x) < \frac{\xi_{1\varepsilon}k^2(d(x))g(\psi(d(x)))}{g(\xi_{1\varepsilon}\psi(d(x)))}\big(c_0\xi_{1\varepsilon}^\rho + \varepsilon\big),$$

$\forall x \in \Omega_{2\delta}$.

Let $\beta \in (0, \delta)$ be arbitrary. We define $\bar{u}_\beta = \xi_{2\varepsilon}\psi(d(x) - \beta)$ for any x with $\beta < d(x) < 2\delta$, and $\underline{u}_\beta = \xi_{1\varepsilon}\psi(d(x) + \beta)$, for any x with $d(x) + \beta < 2\delta$. It follows by $|\nabla d(x)| = 1$ that

$$
\begin{aligned}
&\Delta\bar{u}_\beta(x)-b(x)g\big(\bar{u}_\beta(x)\big)\pm\big|\nabla\bar{u}_\beta(x)\big|^q\\
&\quad=\xi_{2\varepsilon}k^2\big(d(x)-\beta\big)g\big(\psi\big(d(x)-\beta\big)\big)-b(x)g\big(\xi_{2\varepsilon}\big(\psi\big(d(x)-\beta\big)\big)\big)\\
&\qquad-\xi_{2\varepsilon}k'\big(d(x)-\beta\big)\sqrt{2G\big(\psi\big(d(x)-\beta\big)\big)}-\xi_{2\varepsilon}k\big(d(x)-\beta\big)\sqrt{2G\big(\psi\big(d(x)-\beta\big)\big)}\Delta d(x)\\
&\qquad\pm\xi_{2\varepsilon}^q k^q\big(d(x)-\beta\big)\big(2G\big(\psi\big(d(x)-\beta\big)\big)\big)^{q/2}\\
&\quad=\xi_{2\varepsilon}k^2\big(d(x)-\beta\big)g\big(\psi\big(d(x)-\beta\big)\big)\bigg[(1-\tau_0)-\frac{b(x)g(\xi_{2\varepsilon}(\psi(d(x)-\beta)))}{\xi_{2\varepsilon}k^2(d(x)-\beta)g(\psi(d(x)-\beta))}\\
&\qquad-\bigg(\frac{k'(d(x)-\beta)\sqrt{2G(\psi(d(x)-\beta))}}{k^2(d(x)-\beta)g(\psi(d(x)-\beta))}-\tau_0\bigg)-\frac{k(d(x)-\beta)\sqrt{2G(\psi(d(x)-\beta))}}{k^2(d(x)-\beta)g(\psi(d(x)-\beta))}\Delta d(x)\\
&\qquad\pm\frac{\xi_{2\varepsilon}^{q-1}k^q(d(x)-\beta)(2G(\psi(d(x)-\beta)))^{q/2}}{k^2(d(x)-\beta)g(\psi(d(x)-\beta))}\bigg]\\
&\quad\leqslant\xi_{2\varepsilon}k^2\big(d(x)-\beta\big)g\big(\psi\big(d(x)-\beta\big)\big)\bigg[c_0\xi_0^\rho-\big(c_0\xi_{2\varepsilon}^\rho-\varepsilon\big)\\
&\qquad-\bigg(\frac{k'(d(x)-\beta)\sqrt{2G(\psi(d(x)-\beta))}}{k^2(d(x)-\beta)g(\psi(d(x)-\beta))}-\tau_0\bigg)\\
&\qquad-\frac{k(d(x)-\beta)\sqrt{2G(\psi(d(x)-\beta))}}{k^2(d(x)-\beta)g(\psi(d(x)-\beta))}\Delta d(x)\pm\frac{\xi_{2\varepsilon}^{q-1}k^q(d(x)-\beta)(2G(\psi(d(x)-\beta)))^{q/2}}{k^2(d(x)-\beta)g(\psi(d(x)-\beta))}\bigg]\\
&\quad\leqslant 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta\underline{u}_\beta(x)-b(x)g\big(\underline{u}_\beta(x)\big)\pm\big|\underline{u}_\beta(x)\big|^q\\
&\quad=\xi_{1\varepsilon}k^2\big(d(x)+\beta\big)g\big(\psi\big(d(x)+\beta\big)\big)-b(x)g\big(\xi_{1\varepsilon}\big(\psi\big(d(x)+\beta\big)\big)\big)\\
&\qquad-\xi_{1\varepsilon}k'\big(d(x)+\beta\big)\sqrt{2G\big(\psi\big(d(x)+\beta\big)\big)}-\xi_{1\varepsilon}k\big(d(x)+\beta\big)\sqrt{2G\big(\psi\big(d(x)+\beta\big)\big)}\Delta d(x)\\
&\qquad\pm\xi_{1\varepsilon}^q k^q\big(d(x)+\beta\big)\big(2G\big(\psi\big(d(x)+\beta\big)\big)\big)^{q/2}\big)\\
&\quad=\xi_{1\varepsilon}k^2\big(d(x)+\beta\big)g\big(\psi\big(d(x)+\beta\big)\big)\bigg[(1-\tau_0)-\frac{b(x)g(\xi_{1\varepsilon}(\psi(d(x)+\beta)))}{\xi_{1\varepsilon}k^2(d(x)+\beta)g(\psi(d(x)+\beta))}\\
&\qquad-\bigg(\frac{k'(d(x)+\beta)\sqrt{2G(\psi(d(x)+\beta))}}{k^2(d(x)+\beta)g(\psi(d(x)+\beta))}-\tau_0\bigg)-\frac{k(d(x)+\beta)\sqrt{2G(\psi(d(x)+\beta))}}{k^2(d(x)+\beta)g(\psi(d(x)+\beta))}\Delta d(x)\\
&\qquad\pm\frac{\xi_{1\varepsilon}^{q-1}k^q(d(x)+\beta)(2G(\psi(d(x)+\beta)))^{q/2}}{k^2(d(x)+\beta)g(\psi(d(x)+\beta))}\bigg]\\
&\quad\geqslant\xi_{1\varepsilon}k^2\big(d(x)+\beta\big)g\big(\psi\big(d(x)+\beta\big)\big)\bigg[c_0\xi_0^\rho-\big(c_0\xi_{1\varepsilon}^\rho+\varepsilon\big)\\
&\qquad-\bigg(\frac{k'(d(x)+\beta)\sqrt{2G(\psi(d(x)+\beta))}}{k^2(d(x)+\beta)g(\psi(d(x)+\beta))}-\tau_0\bigg)
\end{aligned}
$$

$$-\frac{k(d(x)+\beta)\sqrt{2G(\psi(d(x)+\beta))}}{k^2(d(x)+\beta)g(\psi(d(x)+\beta))}\Delta d(x) \pm \frac{\xi_{1\varepsilon}^{q-1}k^q(d(x)+\beta)(2G(\psi(d(x)+\beta)))^{q/2}}{k^2(d(x)+\beta)g(\psi(d(x)+\beta))}\Bigg]$$
$$\geqslant 0.$$

Similarly to the proof of Theorem 1.1, we can obtain

$$\lim_{d(x)\to 0}\frac{u_\pm(x)}{\psi(d(x))}=\xi_0.$$

By Lemma 2.6, the proof is finished. □

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Appendix A

In this section, we consider existence of solutions to problems ($P_\pm$) and ($Q_\pm$) and prove Lemma 3.5. First we give some preliminary considerations.

Lemma A.1. (See [19, Proof of Theorems 10.1 and 10.2].) *Let $\Psi(x,s,\xi)$ satisfies the following two conditions*:

(D_1) *Ψ is nonincreasing in s for each $(x,\xi)\in(\Omega\times\mathbb{R}^N)$*;
(D_2) *Ψ is continuously differentiable with respect to the ξ variables in $\Omega\times(0,\infty)\times\mathbb{R}^N$.*

If $u,v\in C(\overline{\Omega})\cap C^2(\Omega)$ satisfies $\Delta u+\Psi(x,u,\nabla u)\geqslant\Delta v+\Psi(x,v,\nabla v)$ in Ω and $u\leqslant v$ on $\partial\Omega$, then $u\leqslant v$ in Ω.

Lemma A.2. (From [2].) *Let $b\equiv C_0$ on Ω with $C_0>0$ and g satisfies* (g_1). *If $g'(t)=u^\rho L(t)$, $\rho>0$, L is slowly varying at infinity, then problem* (P_+) *has one solution in $C^2(\Omega)$ for $q\geqslant 0$; and the same statement is true to problem* (P_-) *for $0\leqslant q\leqslant 2$.*

Lemma A.3. (From [24,38].) *Let $g\in C^1[0,\infty)$ satisfies* (g_1) *and* (g_2). *If b satisfies* (b_1), *then problem* (P_0) *has at least one solution $U\in C^2(\Omega)$.*

Existence of solutions to problem ($P_\pm$) *and* ($Q_\pm$)

(I) First we consider the following Dirichlet problem

$$\Delta u\pm|\nabla u|^q=b(x)g(u),\quad u>0,\ x\in\Omega,\ u|_{\partial\Omega}=m,\ m\in\mathbb{N}. \tag{$P_{m\pm}$}$$

By [1, the first Theorem] or [23, Theorem 8.3, p. 301], we see that problem ($P_{m\pm}$) admits one solution $u_{m\pm}\in C^{2+\alpha}(\overline{\Omega})$ for $q\in[0,2]$. Moreover, Lemma 4.1 implies that

$$u_{m+}(x)\leqslant u_{m+1+}(x)\quad\text{and}\quad u_{m-}(x)\leqslant u_{m+1-}(x)\leqslant U(x),\quad\forall x\in\Omega, \tag{A.1}$$

where U is a classical solution to problem (P_0).

Thus $\lim_{x\to\infty} u_{m-}(x) = u_-(x)$, $x \in \Omega$. By standard argument we see that $u_- \in C^2(\Omega)$ is one solution to problem (P_-) for $q \in [0, 2]$.

For $q \in [0, 2]$. Since problem (P_+) has at least one solution in $C^2(\Omega)$ for $b \equiv C_0$ with $C_0 > 0$, we see by the argument in [24,38] that problem (P_+) has at least one solution in $C^2(\Omega)$ for b satisfying (b_1). For $q > 2$, we can see by the argument in [2, pp. 149, 150] that problem (P_+) has at least one solution.

(II) Let us consider the following Dirichlet problem

$$\Delta w - |\nabla w|^q = b(x)g(w), \quad w > 0,\ x \in \Omega/\overline{\Omega}_0, \quad w|_{\partial\Omega} = 1, \quad w|_{\partial\Omega_0} = m,\ m \in \mathbb{N}. \tag{$\mathrm{Q}_{m\pm}$}$$

By the same argument as that in (I), we can prove Lemma 3.5 (II).

The proof is finished.

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